Capillary channels in a gravitational field*

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Abstract

The liquid shape between two vertical parallel plates in a gravity field due to capillary forces is studied. When the physical system achieves its mechanical equilibrium, the capillary surface has mean curvature proportional to its height above a horizontal reference plane and it meets the vertical walls in a prescribed angle. We examine the shapes of these interfaces and their qualitative properties depending on the sign of the capillary constant. We focus to obtain estimates of the size of the meniscus, as for example, its height and volume.

1 Introduction and formulation of the problem

Consider an infinite horizontal reservoir of fluid and let us introduce two vertical parallel plates. The action of capillarity causes that the liquid rises between both plates until a state of mechanical equilibrium. Denote $\mathcal{S}$ the interface liquid-air formed by the fluid between the two plates and whose shape we would like to determined. The fluid surface level at large distance from the plates provides a reference level Π for atmospheric pressure that does not change with perturbations of the fluid surface between the plates. According to the principle of virtual work, the configurations that adopts the liquid between the two plates are characterized by two facts [7]:

1. The mean curvature of $\mathcal{S}$ is proportional to the height of $\mathcal{S}$ with respect to Π (Laplace equation).
2. The angles γ_i with which $\mathcal{S}$ intersects the plates are constant (Young condition). These constants depend only on the materials of the liquid and the plates.

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We also can consider the wetting phenomenon when one spreads out a sufficient amount of liquid on a stripped domain in such way the liquid tends to wet the domain.

Consider (x, y, z) the usual coordinates in Euclidean three-space $\mathbb{R}^3$, $P_i = \{x = a_i\}$ the two vertical planes with $a = a_1 = -a_2 > 0$ and $\Pi = \{z = 0\}$ the horizontal plane. Set $L_i = \Pi \cap P_i$. Denote $\Omega = \{(x, y) \in \mathbb{R}^2; |x| < a\}$ the horizontal strip in Π determined by the two planes, identifying $\mathbb{R}^2$ with Π as usually. Let the height of this capillary free surface $\mathcal{S}$ with respect to Π, assumed nonparametric over Ω, be given by the scalar function $u = u(x, y)$, $(x, y) \in \Omega$. When the capillary and gravity forces are in equilibrium, u satisfies the partial differential equation

$$\text{div } Tu = \kappa u \tag{1}$$

in Ω where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}}.$$

See [7]. Here $\kappa = \rho g/\sigma$ is the capillarity constant with σ, the surface tension, ρ, the difference of densities across the interface $\mathcal{S}$ and g, the gravitational acceleration, with positive and negative sign in the sessile and pendent case respectively. Equation (1) can be interpreted as that the mean curvature H of the surface $z = u(x, y)$ is $\kappa u/2$. The Young condition writes as

$$\nu_i \cdot Tu = \cos \gamma_i \qquad \text{along } L_i \tag{2}$$

where ν_i is the unit exterior normal on L_i. Here γ_i are the contact angles with which $\mathcal{S}$ meets P_i, $i = 1, 2$. The orientation on $\mathcal{S}$ points in the z-positive direction. If the two plates are made with the same materials, $\gamma = \gamma_1 = \gamma_2$. We may normalize so $0 \leq \gamma \leq \pi$. The range $0 \leq \gamma \leq \pi/2$ indicates capillary rise; $\pi/2 < \gamma \leq \pi$ yields capillary fall. The angles γ_i are determined by the volume per unit of length enclosed by the surface. If $\Omega_b = (-a, a) \times (-\frac{b}{2}, \frac{b}{2})$ is a rectangular piece of Ω of length b, an integration of (1) gives

$$\kappa V_b = (\cos \gamma_1 + \cos \gamma_2) b + 4a,$$

where V_b is the enclosed volume by u over the domain Ω_b. Then

$$\cos \gamma_1 + \cos \gamma_2 = \lim_{b \to \infty} \kappa \frac{V_b}{b}.$$

This identity is similar when Ω is a bounded domain, namely,

$$\cos \gamma = \kappa \frac{V_\Omega}{|\partial \Omega|}.$$

When the effect of gravity is ignored, the liquid-air interface is characterized by a constant mean curvature surface. A first example of graph with constant mean curvature on a band is any section of an infinite round cylinder positioned with its axis parallel to P_i. Exactly, this example motivates us to consider that the shape of the surface $\mathcal{S}$ is translationally invariant with respect to the y-coordinate. Our surface will be invariant

by the reflection with respect to the plane $\{y = t\}$ and $\mathcal{S}$ is determined by its intersection with any plane $\{y = t\}$. Then $\mathcal{S}$ is a cylindrical ruled surface.

Classically, it has been studied the capillary problem when the liquid rises in a tube with circular section in such way that our setting reduces then to consider one of the curvature radius is infinite. So, in the literature, the capillary problem studied here it has been considered in the study of the shape of a meniscus facing a vertical plate. As we will see, in the one-dimensional problem, a first integration of (1) is obtained in such way that the solutions can be expressed in terms of elliptic integrals and some estimates of the height of the meniscus have been obtained from these integrals or as limit case of the two-dimensional case [1, 2, 8, 9, 11, 12, 13]. Our interest in this work is twofold. First, we analyze the symmetries of the surface and the shapes adopted depending on the sign of κ. On the other hand, we shall obtain a detailed study of the height of the meniscus, as well as, other estimates on the volume. In this sense, we will follow the same spirit as in [3, 4, 5, 6]. See also [7].

This paper is organized as follows. In Section 2 we describe all cylindrical ruled surfaces whose mean curvature is proportional to its height with respect to Π, making a study of their symmetries. In Section 3, we obtain estimates of the height of the meniscus in the capillary problem. In Section 4 we consider sessile liquid channels with results on existence with respect to the volume enclosed by the channel. Finally in Section 5 we study pendent liquid channels with the main conclusion that the morphologies that will appear differ completely than in the two-dimensional problem.

2 Capillary immersed bands

With more of generality, let S be a cylindrical ruled surface in the space $\mathbb{R}^3$, that is, an immersed surface parametrized as

$$\mathbf{x}(s,t) = \alpha(s) + t\vec{w}, \qquad s \in I, t \in \mathbb{R},$$

where α is a regular planar curve of $\mathbb{R}^3$ defined in some interval I, called the directrix of S and $\vec{w} \in \mathbb{R}^3$, $|\vec{w}| = 1$. In this section we are interested by those cylindrical surfaces that satisfy the capillary equation (1) for some constant $\kappa \neq 0$. Without loss of generality, we assume that α is parametrized by arc length, $\langle \alpha', \vec{w} \rangle = 0$ and the binormal of α in the Frenet trihedron is $-\vec{w}$. Then $H = C_\alpha/2$, where C_α is the curvature of α. Equation (1) implies then

$$C_\alpha(s) = \kappa z(\alpha(s)) + tz(\vec{w})$$

for all s and t. We infer that $z(\vec{w}) = 0$, that is, $\vec{w}$ is a horizontal vector and the rulings of the surface are horizontal straight-lines.

Definition 2.1 *Let $\kappa \neq 0$. A κ-cylindrical surface is a cylindrical ruled surface that locally satisfies the capillary equation (1).*

In particular, each vertical plane orthogonal to the rulings is a plane of symmetry of S. In addition, the angle that makes a such surface with a pair of vertical parallel planes, or a horizontal plane, is constant. For the study of the existence of κ-cylindrical surfaces, we parametrize the surface S as $\mathbf{x}(s,t)=(x(s),t,z(s))$, $s\in I, t\in\mathbb{R}$, where $\alpha(s)=(x(s),z(s))$. We know that

$$x'(s)^2+z'(s)^2=1, \qquad s\in I. \tag{3}$$

Let $\theta(s)$ be the angle between the vectors $\partial/\partial x$ and $\alpha'(s)$. By (3), the equation (1) converts into the O.D.E. system $\mathcal{P}$:

$$x'(s) = \cos\theta(s) \tag{4}$$
$$z'(s) = \sin\theta(s) \tag{5}$$
$$\theta'(s) = \kappa z(s) \tag{6}$$

Theorem 2.2 *The system of ordinary differential equations $\mathcal{P}$ has a unique solution for each initial condition. Moreover the maximal interval of the solution is $\mathbb{R}$.*

Proof: Classical theory yields existence of solutions for each initial data $x(0)=x_0, z(0)=z_0, \theta(0)=\theta_0$. Denote $\mathcal{P}(x_0,z_0,\theta_0)$ the initial value problem for the initial conditions (x_0,z_0,θ_0). If (x,z,θ) is a solution for $\mathcal{P}(x_0,z_0,\theta_0)$, then $(x+a,z,\theta+b)$ is the solution of $\mathcal{P}(x_0+a,z_0,\theta_0+b)$. Thus, we can assume that $x_0=0$ and $\theta_0=0$. Let us denote $\mathcal{P}=\mathcal{P}(z_0)$. In this article, we assume these initial conditions.

For $z(0)=z_0$, we obtain a solution (x,z,θ) defined around $s=0$. It is immediate that $(s,0,0)$ is the solution for $z_0=0$. Assume now $z_0\neq 0$. From (4)-(6),

$$x''=-\theta' z'=-\kappa z z'=-\frac{\kappa}{2}(z^2)'.$$

Then there exists a constant $m\in\mathbb{R}$ such that

$$x'=-\frac{\kappa}{2}z^2+m=\cos\theta.$$

At $s=0$, we have $m=1+\kappa z_0^2/2$. Thus

$$\frac{\kappa}{2}z^2=1-\cos\theta+\frac{\kappa}{2}z_0^2$$

or

$$z(s)^2=z_0^2+\frac{2}{\kappa}\left(1-\cos\theta(s)\right). \tag{7}$$

Therefore z is a bounded function. As a consequence of (7), together with (4)-(6), the first derivatives of x, z and θ are bounded functions and the theory of O.D.E. yields that the solutions can be continued indefinitely. This proves the result. *q.e.d.*

We prove that our κ-cylindrical surfaces have a rich symmetry.

Theorem 2.3 (Symmetry I) *Let $S \subset \mathbb{R}^3$ be a κ-cylindrical surface. Then S is symmetric with respect to any vertical plane parallel to the rulings and that acrosses an extremum of the height of the function z, where $\alpha = (x, z)$.*

Proof: Consider α the directrix curve of S, and we assume the initial data given in Theorem 2.2. It suffices to prove that the trace of α is symmetric with respect to the line $x = x(s_0)$, where s_0 is any value with $\cos\theta(s_0) = \pm 1$. Let $m \in \mathbb{Z}$ be such that $\theta(s_0) = m\pi$. The theorem is proved if for $s \in \mathbb{R}$,

$$\begin{aligned} x(s+s_0) - x(s_0) &= x(s_0) - x(s_0 - s) \\ z(s+s_0) &= z(s_0 - s) \\ \theta(s+s_0) &= 2m\pi - \theta(s_0 - s). \end{aligned}$$

However, these two sets of functions satisfy the same O.D.E. system $\mathcal{P}$ and initial conditions. The uniqueness of solutions of an O.D.E. concludes the proof.

q.e.d.

In a similar way, we have

Theorem 2.4 (Symmetry II) *Let $S \subset \mathbb{R}^3$ be a κ-cylindrical surface and $\alpha = (x, z)$ its directrix. Assume that α accrosses the x-axis at $s = s_0$. Then α is symmetric with respect to the point $\alpha(s_0)$.*

To end this section, we study the symmetries of the surfaces by horizontal translations orthogonal to the rulings. We distinguish two cases depending on the sign of κ.

Theorem 2.5 (Sessile case) *Let S be a κ-cylindrical surface with $\kappa > 0$. Then there exists a horizontal vector $\vec{v}$ orthogonal to the rulings such that S is invariant by the group of translations*

$$G = \{t_n; n \in \mathbb{Z}\}, \qquad t_n(p) = p + n\vec{v}.$$

Moreover, if $\alpha = (x, z)$ is the directrix of S, the function z is a periodic.

Proof: Since that $(x, -z, -\theta)$ is a solution of $\mathcal{P}(-z_0)$ provided (x, z, θ) is the one of the system $\mathcal{P}(z_0)$, we assume $z_0 > 0$. From (7), $z \geq z_0$. Equation (6) implies that θ is strictly increasing and its limit is ∞. Set $T > 0$ the first number such that $\theta(T) = 2\pi$. Again, the uniqueness of solutions in a O.D.E. gives $\alpha(s+T) = \alpha(s) + (x(T), 0)$. This means that the surface is invariant by the group of translations G, with $\vec{v} = (x(T), 0, 0)$.

q.e.d.

Remark 2.6 *As consequence of Theorem 2.5, and because θ is increasing function to infinity, the velocity vector rotates infinite times around the origin.*

From (7) and because $\cos\theta$ takes all the values into the interval $[-1,1]$, we bound the height function z in terms of the lowest height z_0.

Corollary 2.7 *Let S be a κ-cylindrical surface with $\kappa>0$ and denote z the height with respect to the plane Π. Then z satisfies*

$$z_0 \leq z(p) \leq \sqrt{\frac{4}{\kappa}+z_0^2}, \qquad p\in S,$$

and both bounds are achieved.

This means that, fixed κ, the difference between the two extremum of z is bounded by a constant, namely $\sqrt{4/\kappa}$, independent on z_0.

Theorem 2.8 (Pendent case) *Let S be a κ-cylindrical surface with $\kappa<0$. Denote by $\alpha=(x,z)$ its directrix obtained as solution of $\mathcal{P}(z_0)$. Without loss of generality, suppose $z_0<0$.*

1. *If $z_0<-2/\sqrt{-\kappa}$, then there exists horizontal vector $\vec{v}$ orthogonal to the rulings such that S is invariant by the group of translations $G=\{t_n; n\in\mathbb{Z}\}$, with $t_n(p)=p+n\vec{v}$. Moreover z is a periodic function and $z<0$.*
2. *If $z_0=-2/\sqrt{-\kappa}$, then $z_0\leq z<0$, z is strictly increasing and $\lim_{s\to\infty} z(s)=0$.*
3. *If $-2/\sqrt{-\kappa}<z_0<0$, then α is a periodic function. Moreover, z vanishes in a discrete set of point, $z_0\leq z(s)\leq -z_0$ where both extremum are achieved and α is symmetric with respect to any zero of z.*

Proof:

1. Identity (7) implies that z does not vanish and
$$z_0 \leq z(s) \leq -\sqrt{z_0^2+\frac{4}{\kappa}}. \tag{8}$$
From (6), θ is strictly increasing with
$$\theta' \geq -\kappa\sqrt{z_0^2+\frac{4}{\kappa}}.$$
This means that θ increases until infinity. Again, let $T>0$ be the first number where $\theta(T)=2\pi$. The same reasoning as in Theorem 2.5, proves the statement 1. In particular, $\cos\theta(s)$ takes all values in $[-1,1]$ and the bounds in (8) are achieved.
2. From (7), the only zeroes of z occur when $\cos\theta(s)=-1$. If z vanishes at some point, the uniqueness of solutions would imply that $z=0$, which is a contradiction. Thus $z<0$ and $\cos\theta>-1$. Near $s=0$, θ is increasing and the same occurs for

the function z. In addition, $0 \leq \theta(s) < \pi$. Moreover, $z(s) < 0$, $z'(s) > 0$ for $s \in \mathbb{R}$ and

$$\lim_{s\to\infty} z(s) = z_1 \qquad \lim_{s\to\infty} z'(s) = 0,$$

for some number $z_1 \leq 0$. If $z_1 < 0$, by (6) $\theta' > k > 0$, for some constant k and θ would attain the value π. This contradiction yields $z_1 = 0$.

3. First, we prove that z vanishes. Because $z_0 < 0$, the functions z and θ are increasing near $s = 0$. If $z \leq 0$, then

$$\lim_{s\to\infty} z(s) = \delta \qquad \lim_{s\to\infty} z'(s) = 0,$$

for some number $\delta \leq 0$. If $\delta < 0$, (6) implies that θ increases until ∞ and $\cos\theta$ takes all possible value. Equation (7) together with $-2/\sqrt{-\kappa} < z_0$ imply that $z = 0$ at some point. If $\delta = 0$, (6) gives again that either $\theta \to \infty$, which is a contradiction or $\theta \to \theta_0$, for some $\theta_0 < \infty$. Letting $s \to \infty$ in (5), we conclude that $\theta = \pi$, in contradiction with (7) and $-2/\sqrt{-\kappa} < z_0$.

Therefore, z vanishes at some point. We use the Theorem 2.4 to conclude that z is symmetric with respect to any zero of z. Furthermore, z has a minimum at $s = 0$, since $z''(0) = \kappa z_0 > 0$. Then z is a bounded function with $z_0 \leq z \leq -z_0$. Moreover, the same Theorem 2.4 yields $z(2s_0) = -z_0$. Then (7) implies that at $s = 2s_0$ (resp. $s = 0$), z attains a maximum (resp. minimum). The proof finishes using the symmetries of $(x(s), z(s))$ given in Theorem 2.3. Exactly, it follows that

$$\begin{aligned} x(s+4s_0) &= x(s) + x(4s_0) && (9)\\ z(s+4s_0) &= z(s) && (10)\\ \theta(s+4s_0) &= \theta(s) && (11) \end{aligned}$$

q.e.d.

3 Estimates of capillary strips: case $\kappa > 0$

In this section, we consider κ-cylindrical surfaces S that are graphs over the strip Ω of a function u, that is, S is the surface $z = u(x, y)$ that projects simply onto Ω. We will derive estimates for the capillary rise, as for example, the center height u_0 and the outer height $u(a)$. For the two dimensional problem, we refer [5, 6, 7].

Setting $r = x$ and $u(r, y) = u(r)$, Equation (1) becomes

$$\frac{u''(r)}{(1+u'(r)^2)^{3/2}} = \frac{d}{dr}\left(\frac{u'(r)}{\sqrt{1+u'(r)^2}}\right) = \kappa u. \tag{12}$$

Together (12), we consider the initial conditions

$$u(0) = u_0 > 0, \qquad u'(0) = 0. \tag{13}$$

Denote $u = u(r; u_0)$ the dependence of u with respect to the initial condition $u(0) = u_0$. It is immediate then that

1. $u(r; 0) = 0$ and $u(-r; u_0) = u(r; u_0)$.
2. $u(r; u_0) = -u(r; -u_0)$: up a symmetry with respect to the r-axis, the sign of the initial condition u_0 can be prescribed.

According to these properties, we will assume that $u_0 \neq 0$ and that the signs of u_0 and κ agree. Although much of our results are valid with independence on the sign of κ, we restrict to the case that κ is a positive number.

The boundary condition (2) writes now $u'(a) = \cot\gamma$. We know by standard theorems of O.D.E. that there exists such function u defined in some interval around $r = 0$. Put

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}}, \qquad \cos\psi = \frac{1}{\sqrt{1+u'^2}},$$

where ψ is the angle that makes $u(r)$ with the horizontal line at each point $(r, u(r))$. Then (12) takes the form

$$(\sin\psi)' = \kappa u. \tag{14}$$

For $r > 0$ and close to 0,

$$\sin\psi = \kappa \int_0^r u(t)\, dt.$$

As $u_0 > 0$, the integrand is positive near to $r = 0$. Then $\sin\psi > 0$, and so, $u'(r) > 0$. This means that u is increasing provided that u is defined in the maximal interval $(0, R)$, $0 < R \leq \infty$. Multiplying by u' in (12), we have a first integration

$$\frac{1}{\sqrt{1+u'^2}} = -\frac{\kappa}{2}u^2 + c,$$

for some constant c. At $r = 0$,

$$c = 1 + \frac{\kappa}{2}u_0^2.$$

Therefore

$$u' = \sqrt{\frac{4}{\left(2+\kappa(u_0^2 - u^2)\right)^2} - 1} \tag{15}$$

and

$$u(r) = u_0 + \int_0^r \sqrt{\frac{4}{(2+\kappa(u_0^2 - u(t)^2)^2} - 1}\, dt.$$

From (12) and (13), we have $u'' \geq \kappa u \geq \kappa u_0 > 0$. This implies that u' in increasing on r and $u'(R) = \infty$. Equation (15) (or (7)) gives

$$u(R) = \sqrt{\frac{2}{\kappa} + u_0^2}.$$

This means that $R < \infty$ and that the maximal distance between the center and outer height of a κ- cylindrical surface is

$$u(R) - u_0 = \frac{\frac{2}{\kappa}}{u_0 + \sqrt{\frac{2}{\kappa} + u_0^2}}. \tag{16}$$

This was to be expected according to the Remark 2.6 and (7). As a consequence, fixed a capillarity constant $\kappa > 0$ and $u_0 > 0$, the angle of contact γ takes all the values in the range $0 \leq \gamma \leq \pi/2$. More generally, we have from (7)

Corollary 3.1 *Let S be a κ-cylindrical surface given by the profile $u = u(r; u_0)$. If γ is the contact angle with the vertical walls at $r = a$, then*

$$q := u(a) - u(0) = \frac{\frac{2}{\kappa}(1 - \sin\gamma)}{u_0 + \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \sin\gamma)}} < \sqrt{\frac{2}{\kappa}(1 - \sin\gamma)}. \tag{17}$$

Fixing γ, the function $q = q(u_0)$ depending on the initial condition u_0 goes from 0 to $\sqrt{2(1 - \sin\gamma)/\kappa}$, with

$$\lim_{u_0 \to 0} q = \sqrt{\frac{2}{\kappa}(1 - \sin\gamma)}, \qquad \lim_{u_0 \to \infty} q = 0.$$

As u is increasing on r, we bound the integrand in (14) by $u_0 < u(t) < u(r)$ obtaining

$$\kappa u_0 < \frac{\sin\psi(r)}{r} < \kappa u(r). \tag{18}$$

Moreover,

$$\lim_{r \to 0} \frac{\sin\psi(r)}{r} = \kappa u_0.$$

This allows to give the following results on existence

Theorem 3.2 *Let $\kappa > 0$ be a constant of capillarity. Given $2a > 0$, the width of the strip Ω, and $0 \leq \gamma < \pi/2$, a contact angle, there exists a κ-cylindrical surface on Ω that makes a contact angle γ with the plates $P_1 \cup P_2$.*

Proof: The problem reduces to find $u_0 > 0$ such that $u'(a; u_0) = \cot\gamma$, or in terms of the function ψ, that $\sin\psi(a) = \cos\gamma$, where $0 < \cos\gamma \leq 1$. If $u_0 = 0$, we know that $u(r; 0) = 0$. By the continuity on the parameter u_0 for the solutions of (12)-(13),

$$\lim_{u_0 \to 0} \sin\psi(a; u_0) = \sin\psi(a; 0) = 0.$$

If we denote by $R = R(u_0)$ the maximal interval of $u(r; u_0)$, and since $R(0) = \infty$, there exists u_0 close to 0 such that the following holds:

$$R(u_0) > a \qquad \sin\psi(a; u_0) < \cos\gamma.$$

From (18), u_0 cannot take any value, but its supremum is $1/(\kappa a)$. Again, the left inequality in (18) leads to

$$\lim_{u_0 \to 1/(\kappa a)} \sin\psi(a;u_0) = 1.$$

By continuity, there exists $u_0 \in (0, 1/(\kappa a))$, such that the solution $u(r;u_0)$ satisfies $\sin\psi(a) = \cos\gamma$.

q.e.d.

Now, we bound the center height u_0 and the outher height $u(a)$. Consider a lower circular arc $\Sigma^{(1)}$: $u^{(1)}(r)$, centered on the u-axis, with $u^{(1)}(0) = u_0$ and of radius $R_1 = 1/(\kappa u_0)$. Let also $\Sigma^{(2)}$: $u^{(2)}(r)$ be a lower circular arc, centered on the u-axis, with $u^{(2)}(0) = u_0$, and such that

$$\frac{d}{dr}u^{(2)}(a) = \frac{d}{dr}u(a)$$

so that $\Sigma^{(2)}$ meets the vertical plates in the same angle as does the solution surface.

The circular arcs $\Sigma^{(1)}$ and $\Sigma^{(2)}$ can parametrized as

$$u^{(1)}(r) = u_0 + R_1 - \sqrt{R_1^2 - r^2}, \qquad R_1 = \frac{1}{\kappa u_0}. \tag{19}$$

$$u^{(2)}(r) = u_0 + R_2 - \sqrt{R_2^2 - r^2}, \qquad R_2 = \frac{a}{\cos\gamma}. \tag{20}$$

Claim. The three functions satisfy $u^{(1)}(r) < u(r) < u^{(2)}(r)$ in the interval $(0, a]$.

Proof: [of the Claim] By (12), the curvature of $u(r)$ is

$$C_u(r) = \frac{u''(r)}{(1 + u'(r)^2)^{3/2}} = \kappa u(r).$$

Moreover C_u is increasing on r since κ and u' are positive. At $r = 0$, $C_u(0) = \kappa u_0 = C_{u^{(1)}}(0)$ and $\Sigma^{(1)}$ has constant curvature. Because $u(0) = u^{(1)}(0)$, we conclude then

$$\frac{d}{dr}u^{(1)}(r) < \frac{d}{dr}u(r); \qquad u^{(1)}(r) < u(r); \qquad 0 < r < a.$$

For $u^{(2)}$, as $u^{(1)}$ and $u^{(2)}$ are circles and $C_{u^{(2)}}(a) > C_{u^{(1)}}(a)$, then $C_{u^{(2)}}(r) > C_{u^{(1)}}(r)$ for any r. At $r = 0$, $C_{u^{(2)}}(0) > C_u(0)$ and $u^{(2)}(0) = u(0)$. Thus, there exists $\delta > 0$ such that $u^{(2)}(r) > u(r)$ for $0 < r < \delta$. We assume that δ is the least upper bound of such values. By contradiction, suppose that $\delta < a$. As $u^{(2)}(\delta) = u(\delta)$ and $u^{(2)\prime}(\delta) \leq u'(\delta)$, $\psi^{(2)}(\delta) \leq \psi(\delta)$ and

$$\int_0^\delta \frac{d}{dr}\left(\sin\psi(r) - \sin\psi^{(2)}(r)\right) dr = \sin\psi(\delta) - \sin\psi^{(2)}(\delta) := \alpha(\delta) \geq 0. \tag{21}$$

Then there exists $\bar{r} \in (0, \delta)$ such that

$$C_u(\bar{r}) = (\sin\psi)'(\bar{r}) > (\sin\psi^{(2)})'(\bar{r}) = C_{u^{(2)}}(\bar{r}).$$

As $C_u(r)$ is increasing, $C_u(r) > C_{u^{(2)}}(r)$ for $r \in (\bar{r}, a)$. In particular, and using $u'(a) = u^{(2)'}(a)$,

$$0 < \int_\delta^a (C_u(r) - C_{u^{(2)}}(r))dr = \int_\delta^a \frac{d}{dr}\left(\sin\psi(r) - \sin\psi^{(2)}(r)\right)dr = -\alpha(\delta)$$

in contradiction with (21). *q.e.d.*

As conclusion, the circular arcs $\Sigma^{(1)}$, $\Sigma^{(2)}$ lie respectively below and above the solution curve. From the Claim, and (19)-(20), we obtain

$$u_0 + \frac{1}{\kappa u_0} - \sqrt{\frac{1}{\kappa^2 u_0^2} - a^2} < u(a) < u_0 + \frac{a}{\cos\gamma}(1 - \sin\gamma).$$

Together with $u_0 < 1/(a\kappa)$ and (17), we conclude

Theorem 3.3 *Let S be a κ-cylindrical surface where γ denotes the contact angle with the vertical plates P_i and $0 \leq \gamma < \pi/2$. Then the difference value $q = u(a) - u(0)$ satisfies*

$$\frac{1}{\kappa u_0}\left(1 - \sqrt{1 - a^2\kappa^2 u_0^2}\right) < q < \frac{a}{\cos\gamma}(1 - \sin\gamma). \tag{22}$$

$$q > \frac{2a(1-\sin\gamma)}{1 + \sqrt{1 + 2\kappa a^2(1-\sin\gamma)}}. \tag{23}$$

We can compare the upper bound for q in (22) with the one obtained in (17). So, in the case that S is vertical at the walls,

$$\frac{2a}{1+\sqrt{1+2\kappa a^2}} < q < \{a, \sqrt{\frac{2}{\kappa}}\}.$$

Another consequence of the Claim is that it allows to compare the volume per unit of length of $\partial\Omega$ between the cylindrical capillary channel and the sections of horizontal round cylinder determined by $\Sigma^{(1)}$ and $\Sigma^{(2)}$. For this, it suffices with

$$\int_0^a u^{(1)}(r)\ dr < \int_0^a u(r)\ dr < \int_0^a u^{(2)}(r)\ dr. \tag{24}$$

The integral for u can be computed by (12):

$$\int_0^a \kappa u(r)\ dr = \cos\gamma. \tag{25}$$

If we denote

$$F(u_0; R) = a(R + u_0) - \frac{a}{2}\sqrt{R^2 - a^2} - \frac{R^2}{2}\arcsin(\frac{a}{R}),$$

then (24) and (25) imply

$$F(u_0; R_1) < \frac{\cos\gamma}{\kappa} < F(u_0, R_2).$$

Thus, each one of the two above inequalities gives

$$a\left(\frac{1}{\kappa u_0} + u_0\right) - \frac{a}{2}\sqrt{\frac{1}{\kappa^2 u_0^2} - a^2} - \frac{\arcsin(a\kappa u_0)}{2\kappa^2 u_0^2} < \frac{\cos\gamma}{\kappa} \tag{26}$$

$$\frac{\cos\gamma}{\kappa} < \frac{a^2}{\cos\gamma} + au_0 - \frac{a^2\tan\gamma}{2} - \frac{a^2}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right). \tag{27}$$

From (27), we obtain a lower bound for u_0. On the other hand, and since $\partial F/\partial u_0 > 0$, let $u_0^+ > u_0$ be the number such that $F(u_0^+; R_1) = \cos\gamma/\kappa$. As $F(x; R) - ax$ is positive,

$$F(\frac{\cos\gamma}{a\kappa}; R_1) > \frac{\cos\gamma}{\kappa} = F(u_0^+; R_1),$$

and thus

$$u_0^+ < \frac{\cos\gamma}{a\kappa}.$$

Theorem 3.4 *Let S be a κ-cylindrical surface, $\kappa > 0$, given by the profile $u = u(r; u_0)$. If $0 \leq \gamma < \pi/2$ denotes the contact angle with the vertical plates P_i at $r = a$, then*

$$\frac{\cos\gamma}{a\kappa} - \frac{a}{\cos\gamma} + \frac{a\tan\gamma}{2} + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right) < u_0 < u_0^+ < \frac{\cos\gamma}{a\kappa}. \tag{28}$$

The left inequality in (28) extends the one obtained by Laplace for the circular capillary tube [10]. The inequality $u_0 < \cos\gamma/(a\kappa)$ is also a consequence by comparing the slopes of $u^{(1)}$ and u at the point $r = a$: ${u^{(1)}}'(a) < u'(a)$. On the other hand, the combination of inequalities (26) and (27) gives an estimate of u_0 that it is rather cumbersome, even in the case $\gamma = 0$:

$$a\left(\frac{1}{\kappa u_0} + u_0\right) - \frac{a}{2}\sqrt{\frac{1}{\kappa^2 u_0^2} - a^2} - \frac{\arcsin(a\kappa u_0)}{2\kappa^2 u_0^2} < a^2 + au_0 - \frac{\pi a^2}{4}.$$

Now, we bound the outer height $u(a)$. Let us move down the circular arc $\Sigma^{(2)}$ until it meets the solution curve (tangentially) at $(a, u(a))$.

Claim. At the contact point $(a, u(a))$, the arc $\Sigma^{(2)}$ lies below the solution curve u.

Proof: [of de Claim] The argument is similar as in the above Claim. In the new position, we compare the curvatures of u and $\Sigma^{(2)}$: by (18), we have

$$C_u(a) = \kappa u(a) > \frac{\sin\psi(a)}{a} = C_{u^{(2)}}(a).$$

Thus, around the point $r = a$, $u > u^{(2)}$. By contradiction, assume that there is $\delta \in (0, a)$ such that $u^{(2)}(r) < u(r)$ for $r \in (\delta, a)$ and $u^{(2)}(\delta) = u(\delta)$. Since $u'(\delta) \geq u^{(2)\prime}(\delta)$, then $\psi^{(2)}(\delta) \leq \psi(\delta)$. This implies

$$\int_\delta^a (C_{u^{(2)}}(r) - C_u(r))dr = \sin\psi(\delta) - \sin\psi^{(2)}(\delta) \geq 0. \tag{29}$$

Then there would be $\bar{r} \in (\delta, a)$ such that $C_{u^{(2)}}(\bar{r}) - C_u(\bar{r}) > 0$. As $C_u(r)$ is increasing on r, $C_u(r) < C_{u^{(2)}}(r)$ on $(0, \bar{r})$ and hence also throughout $(0, \delta) \subset (0, \bar{r})$. Thus

$$0 < \int_0^\delta (C_{u^{(2)}}(r) - C_u(r))dr = \sin\psi(\delta) - \sin\psi^{(2)}(\delta) \leq 0$$

by (29). This contradiction shows the Claim.

q.e.d.

Let $u^{(3)}$ be the displaced arc $\Sigma^{(2)}$. Then the Claim allows to estimate the value $u(a)$ by

$$\int_0^a u^{(3)}(r)dr < \int_0^a u(r)dr.$$

Recall that the center of $u^{(3)}$ is $u_0 - (u^{(2)}(a) - u(a))$. Then

$$F(u_0 + u(a) - u^{(2)}(a); R_2) < \frac{\cos\gamma}{\kappa}.$$

Theorem 3.5 *With the same notation as in Theorem 3.4, for any* $0 \leq \gamma < \pi/2$ *and* $\kappa > 0$,

$$u(a) < \frac{\cos\gamma}{\kappa a} - \frac{a}{2}\tan\gamma + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right). \tag{30}$$

Corollary 3.6 *For any* $r \in (0, a)$ *and* $0 \leq \gamma < \pi/2$*, we have*

$$\frac{r^2\kappa u_0}{1 + \sqrt{1 - r^2\kappa^2 u_0^2}} < u(r) - u_0 < \frac{a}{\cos\gamma} - \sqrt{\frac{a^2}{\cos^2\gamma} - r^2} \tag{31}$$

and

$$u(a) + \frac{a\sin\gamma}{\cos\gamma} - \sqrt{\frac{a^2}{\cos^2\gamma} - r^2} < u(r) - u_0. \tag{32}$$

Proof: The bounds (31) are consequence of $u^{(1)}(r) < u(r) < u^{(2)}(r)$. The lower bound (32) comes from $u^{(3)}(r) < u(r)$ in $(0, a)$.

q.e.d.

This section ends by obtaining lower estimates for the values u_0 and the difference value $q = u(a) - u(0)$. Since $u' > 0$ in the interval $(0, a)$, we may introduce the inclination angle $\psi = \arctan u'(r)$ as independent variable. We have then

$$\frac{dr}{d\psi} = \frac{\cos\psi}{\kappa u} \qquad \frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}. \tag{33}$$

Simple quadratures then yield again

$$u(\psi) = \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \cos\psi)} \tag{34}$$

obtained in (7). As a consequence, *the difference of squares of the maximum and minimum heights satisfies*

$$u^2(\psi) - u_0^2 = \frac{2}{\kappa}(1 - \cos\psi). \tag{35}$$

and thus, independent of the width of the strip Ω.

As $r\kappa u_0 < \sin\psi$,

Corollary 3.7 *In the range* $0 < \psi \leq \pi/2$ *there holds*

$$u(\psi) < \sqrt{\left(\frac{\sin\psi}{\kappa r}\right)^2 + \frac{2}{\kappa}(1 - \cos\psi)}.$$

Now, the following computations are similar to the case that Ω is a circular disc [4]. Let

$$m = \cos(\psi/2), \qquad p = \sqrt{1 + \kappa(r/m)^2}.$$

The function r/m is increasing in ψ. As

$$u < \frac{\sin\psi}{\kappa r} p,$$

it follows from (33) that $pdr > r\cot\psi d\psi$, that is

$$\frac{\sqrt{m^2 + \kappa r^2}}{mr} dr > \cot\psi d\psi. \tag{36}$$

From (18),

$$\lim_{\psi\to 0} \frac{r(\psi)}{\sin\psi} = \frac{1}{\kappa u_0}.$$

An integration in (36) leads to

Theorem 3.8 *In the range $0 < \psi \leq \pi/2$ there holds*

$$u_0 > \frac{\sin\psi}{2\kappa r}\frac{\kappa}{m}(1+p)e^{1-p}. \tag{37}$$

Theorem 3.9 *There holds always for any $0 \leq \gamma < \pi/2$*

$$\frac{2(1-\sin\gamma)}{\kappa f(\gamma)} < u(a) - u_0 < \frac{a(1-\sin\gamma)}{\cos\gamma},$$

where

$$f(\gamma) = \frac{2\cos\gamma}{\kappa a} - \frac{a\tan\gamma}{2} + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right).$$

Proof: The right inequality is a consequence of (22). For the left one, we know from (35) that

$$u(a) - u_0 = \frac{2(1-\sin\gamma)}{\kappa(u(a)+u_0)}.$$

Then we bound $u(a)$ and u_0 by (28) and (30).

q.e.d.

4 Sessile liquid channels

In this section we study the setting of a liquid deposited over the strip Ω. As $\kappa > 0$, the vertical gravity fields points towards Π. We know that $u = u(r, y)$, $(r, y) \in \Omega$, satisfies div $Tu = \kappa u$. We assume $u(r) = u(r, y)$ and u satisfies the equation (12) with $u(0) = u_0 > 0$. We write (12) in terms of the inclination angle ψ with respect to the r-axis:

$$\frac{dr}{d\psi} = \frac{\cos\psi}{\kappa u} \qquad \frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}. \tag{38}$$

We point out that this angle ψ agrees at the contact between S and Π with the value γ, the angle which the surface meets Π along the boundary. We know from Section 2 that there exists $R > 0$ where u is vertical, that is, $r(\pi/2) = R$. Theorem 2.5 asserts that ψ takes any real number and the solution $u(\psi)$ can be continued as a solution of (12).

Theorem 4.1 *The functions $r(\psi)$ and $u(\psi)$ can be continued throughout the range $0 < \psi < \pi$. Moreover, there exists a value $r_o = \lim_{\psi\to\pi} r(\psi)$, being $r_o > 0$ and the functions $r(\psi)$ and $u(\psi)$ are monotonely increasing in $(0, \pi)$.*

Proof: From (18), we know that $r < 1/(\kappa u_0)$. Denote by $(-)$ and $(+)$ the part of the meniscus defined for $\psi \in (0, \pi/2)$ and $\psi \in (\pi/2, \pi)$ respectively. For $r < R$ and close to R, we have from (14):

$$1 - \sin\psi^-(r) = \kappa\int_r^R u^- dt; \qquad 1 - \sin\psi^+(r) = \kappa\int_r^R u^+ dt.$$

Subtracting, we get

$$\sin\psi^-(r) - \sin\psi^+(r) = \kappa \int_r^R (u^+(t) - u^-(t))\ dt. \tag{39}$$

In particular, for r close to R, $u^+ > u^-$ and hence $\sin\psi^- > \sin\psi^+$. From (38), u^+, u^- are both increasing in ψ as one can continue the solution until $\psi = \pi$. However it is not possible to arrive until $r = 0$, since (39) would imply $0 > -\sin\psi^+(0) = \kappa \int_0^R (u^+(t) - u^-(t))dt > 0$. Thus, $r_o = \lim_{r\to\pi} r(\psi) > 0$.

q.e.d.

The above computation leads to

Corollary 4.2 *Given $\kappa, u_0 > 0$ and $0 < \gamma \leq \pi$, there exists exactly one κ-cylindrical surface given by the profile $u(r; u_0)$ which makes a contact angle γ.*

We now study the behavior of the sessile liquid channel with respect to the volume that encloses. Although our channels have infinite volume, we can consider the density of fluid, that is, the volume per unit of length. If $\Omega_b = (-a, a) \times (-b/2, b/2) \subset \Omega$, the volume of S in Ω_b is

$$2b\left(ru(a) - \int_0^a u(t)\ dt\right).$$

We call the *volume* of S as

$$\mathcal{V} = 2(ru(r) - \int_0^r u(t)\ dt).$$

We write $\mathcal{V}(\psi)$ to denote the dependence on the angle ψ. Using (14),

$$\mathcal{V}(\psi) = 2\left(ru(r) - \frac{\sin\psi}{\kappa}\right). \tag{40}$$

Theorem 4.3 *Let $V > 0$ and $0 < \gamma \leq \pi$. There is exactly one κ-cylindrical surface resting on Π, $\kappa > 0$, with contact angle γ and volume $\mathcal{V} = V$.*

Proof: The function $\mathcal{V}$ is continuously differentiable on u_0: this follows from the standard continuous dependence theorem of the O.D.E. theory.

We first prove existence. By (18), $R < 1/(\kappa u_0)$. Thus $R \to 0$ as $u_0 \to \infty$. We know from Section 3 that the function $u^{(3)}$ lies below u. Hence that a circle of radius R contains the function $u(\psi)$, for $0 < \psi < \pi/2$. Since $\sin\psi^+ < \sin\psi^-$, the same circle contains also the upper part of u, that is, $u(\psi)$, for $\pi/2 < \psi \leq \pi$. Thus $\mathcal{V} < \pi R^2$, which goes to 0 as $u_0 \to \infty$.

Now, let us $u_0 \to 0$. From (37), $r(\gamma; u_0) \to \infty$ for any fixed $0 \le \gamma \le \pi/2$. By (34), $u(\gamma; u_0) > \sqrt{2/\kappa(1-\cos\gamma)}$. Since the surface is convex, $\mathcal{V} \to \infty$ as $u_0 \to 0$. In the case $\gamma > \pi/2$, $\mathcal{V}(\gamma; u_0) > \mathcal{V}(\pi/2; u_0)$ and the same conclusion holds.

Now, let us fix γ and $\mathcal{V}$. By letting u_0 between 0 and ∞, the volume function $\mathcal{V}$ takes all values. The continuity of $\mathcal{V}$ with respect to u_0 gives the existence of a κ-cylindrical surface with prescribed volume $\mathcal{V}$.

The proof of uniqueness is obtained if we prove

$$\dot{\mathcal{V}} := \frac{\partial \mathcal{V}(\psi; u_0)}{\partial u_0} < 0$$

for all $u_0 > 0$ and each fixed ψ in $0 < \psi \le \pi$. From (40),

$$\dot{\mathcal{V}} = 2(\dot{r}\, u + r\, \dot{u}). \tag{41}$$

It is known that

$$\dot{r}\,(0) = 0, \qquad \dot{u}\,(0) = 1.$$

First, we prove the following

Claim.

$$\frac{d\,\dot{\mathcal{V}}}{d\psi} < 0 \quad \text{in } (0, \pi]. \tag{42}$$

Proof: [of the Claim] By using (38),

$$\frac{d\,\dot{\mathcal{V}}}{d\psi} = \frac{2\sin\psi}{\kappa u^2}(\dot{r}\, u - r\, \dot{u}). \tag{43}$$

We shall prove that $\dot{r} < 0$ and $\dot{u} > 0$. Again, (38) yields

$$\frac{d\,\dot{r}}{d\psi} = -\frac{\cos\psi\, \dot{u}}{\kappa u^2}, \qquad \frac{d\,\dot{u}}{d\psi} = -\frac{\sin\psi\, \dot{u}}{\kappa u^2}. \tag{44}$$

As $u_0 > 0$, $\frac{d\dot{r}}{d\psi}(0) = -1/(\kappa u_0^2) < 0$, $\dot{r} < 0$ in an initial interval $J = (0, \delta)$, with $\delta \le \pi$.

On the other hand, $\dot{u} > 0$ for sufficiently small ψ and thus,

$$\frac{d\,\dot{u}}{d\psi} > -\frac{\sin\psi\, \dot{u}}{\kappa u_0^2}.$$

By integrating this expression, we have

$$\dot{u} > \exp\left\{\frac{\cos\psi - 1}{\kappa u_0}\right\}. \tag{45}$$

We conclude $\dot{u} > 0$ and (45) holds in J. From (43),

$$\frac{d\dot{\mathcal{V}}}{d\psi} < 0 \quad \text{in } J. \tag{46}$$

By contradiction, we suppose there exists $0 < \psi_0 < \pi$ such that $\dot{r}(\psi_0) = 0$. Take ψ_0 the first ψ with this property. As $\dot{\mathcal{V}}(0) = 0$, (46) implies that $\dot{\mathcal{V}}(\psi_0) < 0$. Moreover, (45) gives $\dot{u}(\psi_0) > 0$. Now (41) yields

$$\dot{\mathcal{V}}(\psi_0) = 2r(\psi_0)\,\dot{u}(\psi_0) > 0.$$

This contradiction implies that $\dot{r}$ is negative in $(0,\pi)$and the Claim is showed. *q.e.d.*

Hence $\dot{\mathcal{V}} < 0$ in $(0,\pi)$ for any u_0. This proves the uniqueness of Theorem.

q.e.d.

We establish a relation between the volume $\mathcal{V}$ enclosed by a liquid channel with the width of the strip Ω that defines in Π and the contact angle γ. If $0 < \gamma \leq \pi$ is the contact angle, let us denote $a = r(\gamma)$. The formulas that we will obtain are a consequence to compare the liquid channel with the halfcylinders defined by the function $u^{(3)}$ in Section 3.

We know that the function $u^{(3)}$ is tangent to u at the point $(a, u(a))$, $0 \leq \gamma < \pi/2$. We prove that the semicircle determined by $u^{(3)}$ in the halfplane $r > 0$ contains inside the solution curve u, with a single point of contact, namely, $(a, u(a))$. At this point, we compare the curvatures of the curves u and $u^{(3)}$. By (18), $C_u(a) > C_{u^{(3)}}$. Moreover, C_u is increasing on ψ for any $0 < \psi < \pi$:

$$\frac{dC_u}{d\psi} = \kappa\frac{du}{d\psi} = \frac{\sin\psi}{u} > 0$$

by (38). This proves the inclusion property. As a consequence, we can compare the volume $\mathcal{V}$ of the liquid channel with respect to the halfcylinder determined by $u^{(3)}$. Denote $2R > 0$ the maximal width of the liquid channel, that is, where the fluid is vertical at the walls.

Theorem 4.4 *Let S be a κ-cylindrical surface resting on a horizontal plane Π, $\kappa > 0$, and let γ be the angle of contact. Denote $V(\gamma)$ the enclosed volume by S. In the range $0 < \gamma \leq \pi/2$, there holds:*

$$\mathcal{V}(\gamma) < \frac{a^2}{\sin^2\gamma}\left(\gamma - \sin\gamma\cos\gamma\right). \tag{47}$$

If $\pi/2 \leq \gamma \leq \pi$, there holds

$$\mathcal{V}(\gamma) < R^2(\gamma - \sin\gamma\cos\gamma). \tag{48}$$

Proof: For the case $0 < \gamma \leq \pi/2$, we only point that

$$\int_0^a u^{(3)}(r)\ dr = a^2 \cot\gamma + au(a) - \frac{a^2}{\sin^2\gamma}\left(\frac{\gamma}{2} + \frac{1}{2}\sin\gamma\cos\gamma\right).$$

For (48), we consider a halfcircle v centered at $(0, u(R))$ of radius R. It is known that the lower part of this circle lies below u. We parametrize v by the angle with the r-axis in each point. We prove that $v(\gamma) > u(\gamma)$. Fixed $r < R$ and $\pi/2 \leq \psi \leq \pi$ with $u(\psi) = u(r)$, the function v lies above u at r, $v(r) > u(r)$, and, $\sin\psi^+(r) < \sin\phi^+(r)$, where ϕ^+ is the inclination angle of the curve v. As $\sin\phi^+$ decreases as $\phi^+ \to \pi$, $v(\gamma) > u(\gamma)$. Then (48) is a consequence of the computation of the volume of v until $\psi = \gamma$.

q.e.d.

We see a lower bound of the volume.

Theorem 4.5 *With the same notation as in Theorem 4.4, there holds in the range* $0 < \gamma \leq \pi/2$

$$\mathcal{V}(\gamma) > \frac{\gamma - \sin\gamma\cos\gamma}{\kappa^2 u(\gamma)^2}. \tag{49}$$

Proof: Consider the circle

$$v(r) = u_0 + R - \sqrt{R^2 - r^2}, \qquad R = \frac{1}{\kappa u(\gamma)}.$$

The curve v touches tangentially u at $(0, u_0)$. As $C_u(0) = \kappa u_0 < C_v(0) = \kappa u(\gamma)$, $v(r) > u(r)$ for each r where v is defined. If we prove that $v(\gamma) < u(\gamma)$, then the arc v until $\phi = \gamma$ lies above u and this allows to obtain a lower bound for the volume of u.

At the point where v attains the inclination angle $\psi = \gamma$,

$$v(\gamma) = u_0 + \frac{1 - \cos\gamma}{\kappa u(\gamma)}.$$

Then $u(\gamma) > v(\gamma)$ if $u(\gamma) - u_0 > v(\gamma) - u_0$. By using (34), we have to prove

$$\frac{2(1 - \cos\gamma)/\kappa}{u_0 + \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \cos\gamma)}} > \frac{1 - \cos\gamma}{\kappa u(\gamma)},$$

or equivalently,

$$2u(\gamma) > u_0 + \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \cos\gamma)}.$$

But the second summand on the right side is exactly $u(\gamma)$, using (34) again. This proves that $v(\gamma) < u(\gamma)$. Then (49) is a consequence of the computation of the volume enclosed by the function v.

q.e.d.

Now we prove the following inclusion result.

Theorem 4.6 *Let $0 < \gamma \leq \pi/2$ and let S be a κ-cylindrical surface supported on the horizontal plane Π, $\kappa > 0$, and making contact angle γ. Let $\mathcal{V}$ be its volume. Then every κ-cylindrical surface resting on Π and with smaller volume and making the same contact angle can be translated rigidly so that it lies strictly interior into S.*

Proof: Assume that S is given by the solution $u(r; u_0)$, $u_0 > 0$. Consider the solution $u^\delta = u(r; u_0 + \delta)$, with $\delta > 0$. From (14),

$$\sin\psi^\delta - \sin\psi = \kappa \int_0^r (u^\delta - u)\ dt.$$

As $(u^\delta - u)(0) = \delta > 0$, $\sin\psi^\delta > \sin\psi$. By (18) and (14), $(u^\delta - u)' > 0$. It follows that if we move downward the curve u^δ a distance δ, then it lies above the curve u except at the single point $(0, u_0)$.

The result is proved if for any $\delta > 0$, $u^\delta < u + \delta$ at the points where the angle γ is achieved. For given γ,

$$(u^\delta - u - \delta)(\gamma) = \int_0^\delta (\dot{u} - 1)\ du_0, \tag{50}$$

with

$$\dot{u} = \frac{\partial}{\partial u_0} u(\gamma; u_0).$$

Since $\dot{u}(0; u_0) = 1$,

$$\dot{u} = \int_0^\gamma \frac{d\dot{u}}{d\psi}\ d\psi + 1.$$

As we have seen in the proof of Theorem 4.3), $\dot{u} < 0$ and

$$\frac{d\dot{u}}{d\psi} < 0 \qquad 0 < \psi < \gamma.$$

Thus $\dot{u} - 1 < 0$, which implies that the integrand in (50) is negative, proving the result.

q.e.d.

We end the section obtaining new estimates of a sessile liquid channel, with special attention if the contact angle lies in the range $[\pi/2, \pi]$.

Theorem 4.7 *Let S be a κ-cylindrical surface supported on Π and $\kappa > 0$. Suppose that $u = u(\psi)$ is the profile of S, where ψ denote the inclination angle with respect to the r-axis. Then in the range $0 < \psi \leq \pi$ there holds*

$$u(\psi) - u_0 < \sqrt{\frac{2(1 - \cos\psi)}{\kappa}}.$$

In the range, $\pi/2 \leq \psi \leq \pi$,

$$R - r(\psi) < \frac{1}{\sqrt{\kappa}} \left(\sqrt{2} + \log \left(\tan \frac{\pi}{8} \right) \right) - 2\cos\frac{\psi}{2} - \log\left(\tan\frac{\psi}{4} \right).$$

In particular,

$$u(\psi) - u(R) < \frac{\sqrt{2(1-\cos\psi)} - \sqrt{2}}{\sqrt{\kappa}}, \qquad \pi/2 \leq \psi \leq \pi.$$

$$R - r_o < \sqrt{\frac{2}{\kappa}}, \quad u(\pi) - u(R) < \frac{2-\sqrt{2}}{\sqrt{\kappa}}.$$

Proof: By using (34), we estimate $u(\psi)$ from below as

$$u(\psi) > \sqrt{\frac{2}{\kappa}(1-\cos\psi)}.$$

In combination with (38), we obtain,

$$\frac{du}{d\psi} < \frac{\sin\psi}{\sqrt{2\kappa(1-\cos\psi)}},$$

and for $\pi/2 \leq \psi \leq \pi$,

$$\frac{dr}{d\psi} > \frac{\cos\psi}{\sqrt{2\kappa(1-\cos\psi)}}.$$

The proof finishes by integrating the two above inequalities. *q.e.d.*

The bound $R - r(\psi)$ gives the minimum distance for two liquid channels in fixed parallel strips can be without contact. One can imagine that if the amount of liquid is small, the shapes adopted by the liquid channels are graphs. If we increase the volume of fluid, the interfaces leave to be graphs and $\gamma > \pi/2$. Then there exists a critical angle where the channels touch their self. Theorem 4.7 gives an estimate of the distance between each two consecutive hydrophilic strips. Other estimate is the following

Theorem 4.8 *Let S be a κ-cylindrical surface supported on the plane Π and $\kappa > 0$. Assume that the contact angle γ satisfies $\pi/2 \leq \gamma \leq \pi$. Then*

$$\frac{1}{\sqrt{\kappa}} \frac{1-\sin\gamma}{\sqrt{2(1-\cos\gamma)+\kappa u_0^2}} < R - r(\gamma) < \frac{1}{\sqrt{\kappa}} \frac{1-\sin\gamma}{\sqrt{2+\kappa u_0^2}}$$

Proof: Recall that the angle parameter ψ agree with the real contact angle γ with Π. As $\cos\gamma < \cos\psi < 0$, (34) gives

$$\sqrt{u_0^2 + \frac{2}{\kappa}} < u(\psi) < \sqrt{u_0^2 + 2(1-\cos\gamma)/\kappa}.$$

Substituting into (38), we obtain

$$\frac{1}{\sqrt{\kappa}}\frac{\cos\psi}{\sqrt{2+\kappa u_0^2}} < \frac{dr}{d\psi} < \frac{1}{\sqrt{\kappa}}\frac{\cos\psi}{\sqrt{2(1-\cos\gamma)+\kappa u_0^2}},$$

and the result follows by integrating from $\psi = \pi/2$ until $\psi = \gamma$. *q.e.d.*

We can compare with the situation of absence of gravity and pieces of infinite cylinders, whose boundary is $\partial\Omega = L_1 \cup L_2$. Then for $\pi/2 \leq \gamma \leq \pi$, the amount $R - r(\gamma)$ is exactly $(1-\sin\gamma)/(2H)$, where H is the mean curvature of the cylinder.

5 Pendent liquid channels

This section is devoted to the study of κ-cylindrical surfaces when $\kappa < 0$. In Section 2, Theorem 2.8, we have studied its behavior. Let $\alpha = (x, z)$ the directrix of the surface and without loss of generality, we assume $z_0 = z(0) < 0$. We identify $u(r(s); u_0) = z(s)$, where u is a solution of (12) with $u_0 = z_0$. We ask when S is a graph on Π, that is, if α is a graph on the r-axis. Theorem 2.8 yields the necessary condition $z_0 > -2/\sqrt{-\kappa}$. However, in this range of values, it is still possible that S presents vertical points.

Theorem 5.1 *Let S be a κ-cylindrical surface, $\kappa < 0$. Then S is a graph on Π if and only if*

$$-\sqrt{\frac{2}{-\kappa}} < u_0 < 0.$$

In such case, there hold the following properties for the function $u = u(r; u_0)$:

1. *The function u is periodic and it is defined on $\mathbb{R}$.*
2. *u vanishes in an infinite discrete set of points.*
3. *The inflections of u are their zeros.*
4. *$u_0 \leq u(r) \leq -u_0$, u attains the values $\pm u_0$ and they are exactly the only critical points.*

Proof: From (7),

$$\cos\psi = 1 - \frac{\kappa}{2}(z^2 - z_0^2) \geq 1 + \frac{\kappa}{2}z_0^2.$$

Therefore, $(x(s), z(s))$ has not vertical points if and only if $z_0^2 < -2/\kappa$. In such case, $x' = \cos\psi >> 0$ and x increases strictly to infinity. Let $s = x^{-1}(r)$. Using the notation of (9), let $r_T = x(4s_0)$. Then

$$u(r + r_T) = u(x(s) + x(4s_0)) = u(x(s + 4s_0)) = z(s + 4s_0) = z(s) = u(r).$$

This proves that u is a periodic function. Moreover, the derivative of u, $u' = \tan\psi$, is bounded, which implies that u can be extended to $\mathbb{R}$. From (12), the inflections agree with the zeros of u. The rest of properties are a consequence of Theorem 2.8. *q.e.d.*

We write (7) as

$$u(\psi)^2 - u_0^2 = \frac{2}{\kappa}(1-\cos\psi). \tag{51}$$

The case

$$u_0 = -\sqrt{\frac{2}{-\kappa}}$$

can be treated as above, except that in a discrete set of points, u is vertical. Moreover, (51) implies that these vertical points are the zeros of u. In general, we can estimate the initial interval where one can define a solution u of (12) without vertical points.

Lemma 5.2 *Consider $u = u(r; u_0)$ the solution of (12)-(13). Then u can be continued at least until the value $r = 1/(\kappa u_0)$. Furthermore, $\sin\psi < \kappa u_0 r$*

Proof: Since $(\sin\psi)' = \kappa u$, the function $\sin\psi$ is strictly increasing on r whenever u is negative. Then for $r > 0$ close to $r = 0$, $\sin\psi = u'/\sqrt{1+u'^2}$ is positive. As conclusion, u is increasing on r near to 0 and the expression

$$\sin\psi = \kappa \int_0^r u(t)\ dt$$

can be bounded in both sides by the values u_0 and $u(r)$. Then

$$\kappa u(r) < \frac{\sin\psi}{r} < \kappa u_0, \tag{52}$$

and hence

$$\sin\psi < \kappa u_0 r = 1.$$

This means that $\psi < \pi/2$.

q.e.d.

In general, for pendent liquid channels, we can say more about the vertical points.

Theorem 5.3 *Let α the directrix of a κ-cylindrical surface, $\kappa < 0$, such that the initial condition $z(0) = z_0$ satisfies*

$$-\frac{2}{\sqrt{-\kappa}} < z_0 < -\sqrt{\frac{-2}{\kappa}}. \tag{53}$$

Then

1. *α presents exactly four vertical points in each arc of α determined by its period.*
2. *Each of these points lies in the segment of α between one extremum and one zero of z.*
3. *The height of the vertical points is $\pm\sqrt{z_0^2+2/\kappa}$.*

Proof: By the symmetries of α, it suffices to prove that between $s=0$ and the first time s_0 where α intersects the r-axis, there exists exactly one vertical point. Since $\theta'(s)=\kappa z(s)$, the function θ is increasing on r in the interval $(0,s_0)$. We know that θ attains the value $\theta=\pi/2$, the first vertical point, at some point s^*, with $s^*<s_0$. By using again (51) and (53), θ does not reach the value $\theta=\pi$. Thus there exists a unique vertical point. By (51), the height at $s=s^*$ is $-\sqrt{z_0^2+\frac{2}{\kappa}}$. *q.e.d.*

When α begins from $s=0$, α is a graph on the r-axis until that α is vertical. It is possible to determine the region where this first vertical point occurs. To this end, one can carry as in the case of pendent liquid drops. We refer [3] and [7, Ch. 4.6]. The main argument is a "Comparison Lemma" that compares u with circular arcs and the hyperbolas $ru<1/(2\kappa)$ and $ru<1/\kappa$. For example, one can show that *the directrix α, in the initial region $z<0$, does not enter the region $ru\leq 1/\kappa$.* We omit the details.

We summarize then the behavior of u, a solution of (12), with $u_0>-\sqrt{-2/\kappa}$. After $r=0$, u increases on r until that it touches the r-axis at some point R. Theorem 2.4 says that u is symmetric with respect to the point $(R,0)$. Thus, u increases until the value $r=2R$, where u takes the value $-u_0$. Again, the symmetry of u with respect to the line $r=2R$ implies that u decreases until to arrive at $r=4R$ to reach the value u_0. From this position, the curve u repeats the same behavior by the periodicity of u (recall that the period is $R_T=4R$, see Theorem 5.1). The next result gives an estimate of the value $r=R$ in the sense that, fixed the constant κ, the first zero of u remains bounded in some interval, independent on the initial value u_0.

Theorem 5.4 *Let $\kappa<0$ and $u(r;u_0)$ a solution of (12)-(13) with*

$$-\sqrt{\frac{-2}{\kappa}}<u_0<0. \tag{54}$$

Then

$$\frac{1}{\sqrt{-2\kappa}}<R<\sqrt{\frac{-2e}{\kappa}}. \tag{55}$$

Proof: Since $u(R)=0$, (51) implies that $\cos\psi_R=1+\kappa u_0^2/2$. From (52),

$$R>\frac{\sin\psi(R)}{\kappa u_0}=\frac{1}{\kappa u_0}\sqrt{1-\cos^2\psi_R}=\frac{-1}{2\kappa}\sqrt{-4\kappa-\kappa^2u_0^2}.$$

The left side in (55) is then a consequence of this inequality and (54). Now, we show the right inequality in (55). In the region where $u<0$, $\sin\psi$ is increasing on r. Let us fix *a*

such that $0 < a < R$. Then $\sin\psi(r) > \sin\psi(a)$. As $u'(r) = \tan\psi$ and $\sin\psi < \tan\psi$, we have

$$u' = \tan\psi > \sin\psi \geq \frac{a}{r}\sin\psi(a).$$

A simple integration between a and R gives

$$R < a\exp\left\{\frac{-u(a)}{\sin\psi(a)}\right\}.$$

Again (52) leads to

$$R < a\exp\left\{\frac{-1}{\kappa a^2}\right\}. \tag{56}$$

Since this holds for every $a < R$ and the function on a on the right side of (56) attains a minimum at $a = \sqrt{-2/\kappa}$, we obtain the desired estimate. *q.e.d.*

Following the same steps as in [3, 7], one could improve the upper bound for R.

We analyze the case

$$-\frac{2}{\sqrt{-\kappa}} < z_0 < -\sqrt{\frac{-2}{\kappa}}. \tag{57}$$

Theorem 5.5 *Let $\alpha = (x(s), z(s))$ be the directrix of a κ-cylindrical surface S. Assume that z_0 satisfies (57). Then there exist numbers z_1, z_2, with*

$$-\frac{2}{\sqrt{-\kappa}} < z_2 < z_1 < -\frac{\sqrt{2}}{\sqrt{-\kappa}}$$

and the following properties hold:

1. *If $z_1 < z_0$, then α has not double points and x goes to ∞.*
2. *If $z_0 = z_1$, then α has double points, where α tangentially meets itself at these points, α lies in $\{x \geq 0\}$, and x goes to ∞.*
3. *If $z_2 < z_0 < z_1$, α has double points, meeting at these points transversally and x goes to ∞.*
4. *If $z_0 = z_2$, then α is a closed curve with self intersection at the origin.*
5. *If $z_0 < z_2$, α has double points, where α meets itself transversally and x goes to $-\infty$.*

Proof: Denote $r(\pi/2)$ and $r(\psi_0)$ the abcisas of the first vertical point and the first point which α meets the r-axis. We know that $r(\pi/2) > 0$.

Claim 1. $r(\pi/2) \leq \frac{\pi}{2\sqrt{-2\kappa}}$, independent of the value z_0.

Proof: [of the Claim 1] By (57), $z_0^2 > -2/\kappa$. For each $0 \le \psi \le \psi/2$ and by using (51), we obtain

$$\kappa u(\psi) > -\kappa\sqrt{\frac{-2\cos\psi}{\kappa}}.$$

Since $\cos\psi > 0$, we have from (38) that

$$\frac{dr}{d\psi} < \frac{\sqrt{\cos\psi}}{\sqrt{-2\kappa}} < \frac{1}{\sqrt{-2\kappa}}.$$

Integrating from $\psi = 0$ to $\psi = \pi/2$, we show the Claim 1.

q.e.d.

Claim 2. There exists a continuous function $\varphi = \varphi(z_0)$ strictly decreasing on z_0 such that $r(\psi_0(z_0)) < \varphi(z_0)$, and

$$\lim_{z_0 \to -2/\sqrt{-\kappa}} \varphi(z_0) = -\infty.$$

Proof: [of the Claim 2.] Consider $\psi \in [\pi/2, \psi_0]$. Since $u(\psi_0) = 0$, by (7), we have

$$1 - \cos\psi_0 = -\frac{\kappa}{2} z_0^2.$$

This proves that as $z_0 \to -\frac{2}{\sqrt{-\kappa}}$, the angle ψ_0 which the directrix α meets the r-axis goes to $\psi = \pi$. As $z_0^2 < -4/\kappa$, again (51) leads to

$$u(\psi) \ge -\sqrt{\frac{-2(1+\cos\psi)}{\kappa}}.$$

Then (38) implies

$$\frac{dr}{d\psi} < \frac{1}{\sqrt{-2\kappa}} \frac{\cos\psi}{\sqrt{1+\cos\psi}}.$$

Integrating from $\psi = \pi/2$ until $\psi = \psi_0$, we obtain

$$r(\psi_0) - r(\pi/2) < \frac{1}{\sqrt{-2\kappa}} \int_{\pi/2}^{\psi_0} \frac{\cos\psi}{\sqrt{1+\cos\psi}} d\psi.$$

An integration gives

$$r(\psi_0) - r(\pi/2) < \frac{2}{\sqrt{-\kappa}} \left(\sin(\psi_0/2) - \operatorname{arctanh}(\tan(\psi_0/4)) - \frac{\sqrt{2}}{2} + \operatorname{arctanh}(\tan(\pi/8)) \right).$$

From the Claim 1, $r(\pi/2)$ is bounded. Then, up a constant C,

$$r(\psi_0) < \frac{2}{\sqrt{-\kappa}} \left(\sin(\psi_0/2) - \operatorname{arctanh}(\tan(\psi_0/4))\right) + C := \varphi(z_0).$$

Finally, because $\psi_0 \to \pi$, we have $\varphi \to -\infty$ as $z_0 \to -2/\sqrt{\kappa}$.

q.e.d.

We know from (51) that if $u_a < u_b$, then $\cos\psi_0(u_a) < \cos\psi_0(u_b)$. Moreover,

Claim 3 . The function $r(\psi_0(z_0))$ is strictly decreasing on z_0.

Proof: Consider $u_a < u_b$. A reasoning similar as in Theorem 4.6 proves that if $\delta < 0$, $u(r; z_0 + \delta) + \delta > u(r; z_0)$ (by the Lemma 5.2, $r(\pi/2; z_0 + \delta) < r(\pi/2; z_0)$). This shows that if we move upwards α_a until to arrive the point $(0, u_b)$, α_a lies over α_b at least until the first vertical point of α_a. Then α_a, in the new position, lies over α_b at least until that both α_a and α_b meet the r-axis. If $\bar{r}$ is the x-coordinate of the point where (the displaced) α_a intersects the r-axis in the first time, we have,

$$r(\psi_0(u_a)) < \bar{r} < r(\psi_0(u_b)).$$

q.e.d.

Now, we sketch the proof of the Theorem and we omit the details. Take z_0 varying from $z_0 = -\sqrt{-2/\kappa}$ until $z_0 = -2/\sqrt{-\kappa}$. Let z_1 and z_2 be the unique numbers, $z_1 < z_2$ such that, in the notation of (9), there hold

$$x(2s_0(z_1)) = 0 \quad \text{and} \quad r(\psi_0(z_2)) = 0.$$

The existence is given by the Claim 2 and the uniqueness by the Claim 3. By (9), the direction, left or right, that takes α depends on the sign of $x(4s_0)$. The critical time occurs when $x(4s_0) = x(s_0) = 0 = r(\psi_0(z_2))$, where α is a closed curve. Moreover, since z is increasing in $(0, 2s_0)$ (and z goes from z_0 to $-z_0$), α does not intersect itself. Then the results follow using the symmetries of the directrix according the Theorems 2.3 and 2.4.

q.e.d.

Remark 5.6 *The results obtained here show the contrast of behavior between pendent liquid channels and pendent liquid rotational drops. In the latter setting, for the values of $z_0 < 0$ where α presents vertical points, the number of vertical points goes increasing as $z_0 \to -\infty$ [3]. However, in our case, the periodicity of the curve α simplifies the scene.*

References

[1] V. G. Babak, Thermodynamics of free and of interacting curved interfaces in liquid films, Russian Chem. Rev. **62** (1993), 703–727.

[2] G. Bakker, Kapillarität und oberflächenspannung. In: Handbuch der Experimentalphysik, Band 6. Akademische Verlagsgesellschaft, Leipzing, 1928.

[3] P. Concus, R. Finn, The shape of a pendent liquid drop, Philos. Trans. Roy. Soc. London Ser. A **292** (1979), 307–340.

[4] R. Finn, The sessile liquid drop I: Symmetric Case. Pacific J. Math. **88** (1980), 541–587.

[5] R. Finn, On the Laplace formula and the meniscus height for a capillary surface, Z. Angew. Math. Mech. **61** (1981), 165–173.

[6] R. Finn, Addenda to my paper "On the Laplace formula and the meniscus height for a capillary surface", Z. Angew. Math. Mech. **61** (1981), 175–177.

[7] R. Finn, Equilibrium Capillary Surfaces, Springer-Verlag, Berlin, 1986.

[8] P. de Gennes, F. Brochard-Wyart, D. Queré, Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves, Springer, New York, 2004.

[9] W. Heywang, Zur Stabilitat senkrechter Schmelzzonen, Z. Naturforsch **11a** (1956), 238–243.

[10] P. S. Laplace, Traité de mécanique céleste; suppléments au Livre X, 1805 and 1806 resp. in Oeuvres Comples vol. 4, Gauthier-Villars, Paris.

[11] A. W. Neumann, Contact angles and their temperature dependence: thermodynamic status, measurement, interpretation and application, Adv. Coll. Interf. Sci. **4** (1974), 105–191.

[12] D. Siegel, Height estimates for capillary surfaces, Pacific J. Math. **88** (1980), 471–516.

[13] T. Vogel, Symmetric unbounded liquid bridges, Pacific J. Math. **103** (1982), 205–241.